Intrinsic random functions and universal kriging on the circle

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ABSTRACT

Intrinsic random functions (IRF) provide a versatile approach when the assumption of second-order stationarity is not met. Here, we develop the IRF theory on the circle with its universal kriging application. Unlike IRF in Euclidean spaces, where differential operations are used to achieve stationarity, our result shows that low-frequency truncation of the Fourier series representation of the IRF is required for such processes on the circle. All of these features and developments are presented through the theory of reproducing kernel Hilbert space. In addition, the connection between kriging and splines is also established, demonstrating their equivalence on the circle.

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1. Introduction

When a random process is considered on a circle, it is often assumed to be second-order stationary (or stationary for short in the paper), that is, the mean of the process is constant over the circle and the covariance function at any two points depends only on their angular distance (Yaglom, 1961; Roy, 1972; Roy and Dufour, 1974; Dufour and Roy, 1976; Wood, 1995; Gneiting, 1998). While stationarity is commonly assumed, it is often considered to be unrealistic in practice. In Euclidean spaces, a large class of non-stationary phenomena may be represented through intrinsic random functions (IRF, Matheron, 1973; Cressie, 1993; Chilès and Delfiner, 2012). The properties of IRF in other spaces, such as the circle or sphere, are not widely known. In this paper, the theory of IRF on the circle is developed, where we find that instead of differential operations, truncation of the Fourier series representation becomes essential for IRF on the circle. This can be presented in the context of the reproducing kernel Hilbert space (RKHS, Aronszajn, 1950; Wahba, 1990a). We formally make such a connection and further relate universal kriging with the smoothing formula in RKHS. Based on this approach, we are able to demonstrate the equivalence between splines and kriging on the circle.

2. IRF and RKHS

A key component for IRF is the allowable measure. Based on Matheron (1973) and Chilès and Delfiner (2012, Chapter 4), a discrete measure $\lambda = \sum_{i=1}^{m} \lambda_i \delta(t_i)$ on a unit circle S, where $t_i \in S$, $\lambda_i \in \mathbb{R}$ and $\delta(\cdot)$ is the Dirac measure, is allowable at the order of an integer κ ($\kappa \geq 0$) if it annihilates all trigonometric functions of order $k < \kappa$. That is,

$$\sum_{i=1}^{m} \lambda_i \cos(kt_i) = \sum_{i=1}^{m} \lambda_i \sin(kt_i) = 0, \quad 0 \leq k < \kappa. \tag{1}$$

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We call Λ_κ the class of such allowable measures. Clearly $\Lambda_{\kappa+1} \subset \Lambda_\kappa$. In addition, for $\lambda \in \Lambda_\kappa$, the translated measure $\iota_t\lambda = \sum_{i=1}^{m} \lambda_i \delta(t_i + t), t \in S$ remains in Λ_κ. This can be easily seen from the elementary trigonometric identities (also see Matheron, 1979; Chilès and Delfiner, 2012). For any function $f(\cdot)$ on S, we define $f(\lambda) = \sum_{i=1}^{m} \lambda_i f(t_i)$.

In this paper, we consider a random process $\{Z(t), t \in S\}$ on a unit circle with finite second moment and being continuous in quadratic mean. By Yaglom (1961) and Roy (1972), the process can be expanded in a Fourier series which is convergent in quadratic mean:

$$Z(t) = Z_0 + \sum_{n=1}^{\infty} (Z_{n,c} \cos nt + Z_{n,s} \sin nt), \tag{2}$$

where $Z_0 = 1/(2\pi) \int_S Z(t)dt$, $Z_{n,c} = (1/\pi) \int_S Z(t) \cos nt dt$, and $Z_{n,s} = (1/\pi) \int_S Z(t) \sin nt dt$.

Definition 2.1. For an integer $\kappa (\kappa \geq 0)$, the random process in (2) is called an IRFκ if for any $\lambda \in \Lambda_\kappa$, the process

$$Z_\lambda(t) = Z(\iota_t\lambda) = \sum_{i=1}^{m} \lambda_i Z(t_i + t)$$

is stationary with respect to $t \in S$ and has a zero mean.

To characterize such a circular IRFκ, we denote

$$Z_\kappa(t) = \sum_{n=\kappa}^{\infty} (Z_{n,c} \cos nt + Z_{n,s} \sin nt),$$

as its low-frequency truncated process and so we have the following lemma.

Lemma 2.1. *A random process given by (2) is an IRFκ if and only if its low-frequency truncated process $Z_\kappa(t)$ is stationary and has a zero mean.*

Proof. In the Fourier expansion (2), the lower trigonometric functions will be annihilated by $\lambda \in \Lambda_\kappa$, which implies

$$Z(\iota_t\lambda) = \sum_{n=\kappa}^{\infty} \left(Z_{n,c} \sum_{i=1}^{m} (\lambda_i \cos nt_i \cos nt - \lambda_i \sin nt_i \sin nt) + Z_{n,s} \sum_{i=1}^{m} (\lambda_i \sin nt_i \cos nt + \lambda_i \cos nt_i \sin nt) \right).$$

Denote $\lambda_{n,c} = \sum_{i=1}^{m} \lambda_i \cos nt_i$, $\lambda_{n,s} = \sum_{i=1}^{m} \lambda_i \sin nt_i$ and

$$Y_{n,c} = Z_{n,c}\lambda_{n,c} + Z_{n,s}\lambda_{n,s}, \qquad Y_{n,s} = -Z_{n,c}\lambda_{n,s} + Z_{n,s}\lambda_{n,c}, \quad n = \kappa, \kappa + 1, \ldots, \tag{3}$$

we have

$$Z(\iota_t\lambda) = \sum_{n=\kappa}^{\infty} (Y_{n,c} \cos nt + Y_{n,s} \sin nt).$$

Yaglom (1961, Theorem 5) shows that a random process (2) on the circle is stationary if and only if its Fourier coefficients are uncorrelated random variables. The lemma can be directly obtained based on this and the linear mapping between the coefficients of $(Z_{n,c}, Z_{n,s})$ and $(Y_{n,c}, Y_{n,s})$ in (3). □

Remark 2.1. In Euclidean spaces, the IRF is associated with differential operations (Matheron, 1973; Chilès and Delfiner, 2012). For example, a differentiable IRFκ on a real line is characterized as that its $(\kappa + 1)$ derivative is stationary. Lemma 2.1 indicates that for circular processes, the low-frequency truncation operation replaces differential operations and leads to stationarity. This observation also has important implications for splines on the circle, which is addressed in Section 4.

Remark 2.2. It is clear that an IRF0 on the circle is the conventional stationary process. Note that this is slightly different from what has been defined in the Euclidean spaces, where IRF(−1) is usually a stationary process. For the rest of this paper, we assume $\kappa \geq 1$ for notational simplicity.

Remark 2.3. Based on Lemma 2.1, for an IRFκ process $Z(t)$, the random process

$$Z^*(t) = Z(t) + A_0 + \sum_{n=1}^{\kappa-1} (A_{n,c} \cos nt + A_{n,s} \sin nt),$$

where $A_0, A_{n,c}, A_{n,s}, n = 1, \ldots, (\kappa - 1)$ are random variables, is clearly also an IRFκ. These two processes $Z(t)$ and $Z^*(t)$ share the same truncation process $Z_\kappa(t)$, with $Z^*(\lambda) = Z(\lambda)$, for any $\lambda \in \Lambda_\kappa$. Similar to the discussion in Chilès and Delfiner (2012, Section 4.4.2), these functions form an equivalent class.

To obtain the covariance function of $Z_\kappa(t)$ of an IRFκ, based on Yaglom (1961) or Roy (1972), we denote

$$E(Z_{n,c}Z_{m,c}) = E(Z_{n,s}Z_{m,s}) = I(n, m)\gamma_n,$$

with $\gamma_n > 0$, $n \geq \kappa$, and $\sum_{n=\kappa}^{\infty} \gamma_n < \infty$. Here $I(n, m) = 1$ if $n = m$ and 0 otherwise. This leads to

$$\text{cov}(Z_\kappa(x), Z_\kappa(y)) = \sum_{n=\kappa}^{\infty} \gamma_n \cos n(x - y) := \phi(x - y), \quad x, y \in S.$$

This covariance function $\phi(\cdot)$ plays an essential role in our paper, and is named as the *intrinsic covariance function* of the IRFκ.

Remark 2.4. Another component in IRF is the generalized covariance function. For an IRFκ on the circle, it is clear that $\phi(\cdot)$ is a candidate of the generalized covariance function. By the annihilation property of allowable measures, any function which is the sum of $\phi(\cdot)$ and a linear combination of lower trigonometric functions up to order $\kappa - 1$ can also be a generalized covariance function. A basic property of the generalized covariance function in Euclidean spaces is its conditional positive definiteness. Such a property (Levesley et al., 1999) seems less important on the circle. One can clearly show that $\phi(\cdot)$ is conditional positive definite of order κ, and positive definite simultaneously (Schoenberg, 1942). For example, it is noted by Yaglom (1961) that the space of valid variograms coincides with the space of valid covariances.

RKHS was introduced in Aronszajn (1950), and is popularly used in the spline literature (see, for example Wahba, 1990a). Taijeron et al. (1994) and Levesley et al. (1999) have studied RKHS in the context of spline interpolation and smoothing on the circle and sphere. Here, we formally establish the connection between IRF and RKHS on the circle. First, given an IRFκ process and its intrinsic covariance function $\phi(\theta) = \sum_{n=\kappa}^{\infty} \gamma_n \cos n\theta$, $\gamma_n > 0$, one can define a function space X_κ on S, following Levesley et al. (1999)

$$X_\kappa = \left\{ f(t) = a_0 + \sum_{n=1}^{\infty} (a_{n,c} \cos nt + a_{n,s} \sin nt), t \in S : \sum_{n=\kappa}^{\infty} \frac{1}{\gamma_n} (a_{n,c}^2 + a_{n,s}^2) < \infty \right\}.$$

For $f, g \in X_\kappa$, a semi-inner product is defined

$$\langle f, g \rangle_\kappa = \sum_{n=\kappa}^{\infty} \frac{1}{\gamma_n} (a_{n,c,f} a_{n,c,g} + a_{n,s,f} a_{n,s,g}). \tag{4}$$

There is a nil space for this semi-inner product $N = \text{span}\{1, \cos t, \sin t, \ldots, \cos(\kappa - 1)t, \sin(\kappa - 1)t\}$. Denote $l = \dim(N)$, and it is clear that $l = 2\kappa - 1$. Let $\{\tau_1, \ldots, \tau_l \in S\}$ be a set of distinct points such that for every $p(\cdot) \in N$, if $p(\tau_\nu) = 0$ for all $\nu = 1, \ldots, l$, then $p \equiv 0$. Then, the inner product

$$\langle f, g \rangle = \sum_{\nu=1}^{l} f(\tau_\nu) g(\tau_\nu) + \langle f, g \rangle_\kappa$$

is well defined and X_κ can be shown to be complete with respect to the norm induced by this inner product (Levesley et al., 1999). In addition, there exist $p_1(t), \ldots, p_l(t) \in N$, such that $p_\nu(\tau_\mu) = I(\nu, \mu)$ for $1 \leq \nu, \mu \leq l$. As discussed in Levesley et al. (1999), the space X_κ is a Hilbert function space in which point evaluations are continuous linear functionals. Therefore, for $x, y \in S$, there exists a reproducing kernel

$$H(x, y) = \phi(x - y) - \sum_{\nu=1}^{l} (\phi(x - \tau_\nu) p_\nu(y) + \phi(y - \tau_\nu) p_\nu(x))$$

$$+ \sum_{\nu=1}^{l} \sum_{\mu=1}^{l} \phi(\tau_\nu - \tau_\mu) p_\nu(x) p_\mu(y) + \sum_{\nu=1}^{l} p_\nu(x) p_\nu(y). \tag{5}$$

Proposition 2.1. *$H(x, y)$ is positive definite for $x, y \in S$.*

Proof. By the expansion of $\phi(\theta) = \sum_{n=\kappa}^{\infty} \gamma_n \cos(n\theta)$, we have

$$H(x, y) = \sum_{\nu=1}^{l} p_\nu(x) p_\nu(y) + \sum_{n=\kappa}^{\infty} \gamma_n \left(\cos(nx) - \sum_{\nu=1}^{l} \cos(n\tau_\nu) p_\nu(x) \right) \left(\cos(ny) - \sum_{\nu=1}^{l} \cos(n\tau_\nu) p_\nu(y) \right)$$

$$+ \sum_{n=\kappa}^{\infty} \gamma_n \left(\sin(nx) - \sum_{\nu=1}^{l} \sin(n\tau_\nu) p_\nu(x) \right) \left(\sin(ny) - \sum_{\nu=1}^{l} \sin(n\tau_\nu) p_\nu(y) \right).$$

The positive definiteness of $H(x, y)$ can be obtained through the symmetry in x and y along with $\gamma_n > 0$ for all $n \geq \kappa$. $\quad \square$

Based on Proposition 2.1, there exists a Gaussian random process that is an IRFκ with $H(x, y)$ as its covariance function. Hence, we formally establish the connection between IRF and RKHS on the circle as the following proposition.

Proposition 2.2. *For an IRFκ on the circle, there exists a corresponding RKHS. Its converse is also true.*

Remark 2.5. One can easily verify the reproducing property of this kernel (Taijeron et al., 1994; Light and Wayne, 1995; Levesley et al., 1999). The different choice of the distinct points $\{\tau_1, \ldots, \tau_l\}$ will alter the form of the reproducing kernel with $\phi(\cdot)$ unchanged since the terms containing $p_\nu(\cdot)$ will all be annihilated by $\lambda \in \Lambda_\kappa$.

3. Universal kriging

Universal kriging associated with IRF is widely used in spatial statistics (Cressie, 1989; Chilès and Delfiner, 2012). In this Section, we will discuss universal kriging on the circle. Let $Z(t)$ be an IRFκ with an intrinsic covariance function $\phi(\cdot)$ and the mean function $\mathrm{E}(Z(t)) = \sum_{\nu=1}^{l} \beta_\nu q_\nu(t)$, where $\beta_\nu, \nu = 1, \ldots, l$ are coefficients and $\mathrm{span}\{q_1(t), \ldots, q_l(t)\} = N$. Here $q_\nu(\cdot)$ can be the elementary lower trigonometric functions or $\{p_\nu(t)\}_{\nu=1}^{l}$ given in Section 2. Assume that the data $\{(t_i, y_i), i = 1 \ldots, n\}, n \geq l$, are observed from this IRFκ with measurement error

$$Y(t) = Z(t) + \epsilon(t), \quad t \in S,$$

where $\epsilon(\cdot)$ is a white noise process with mean zero and variance σ^2 that is uncorrelated with the process $Z(\cdot)$.

To obtain the best linear unbiased estimator at $t_0 \in S$, the universal kriging is commonly used, where the linear estimator is

$$\hat{Z}(t_0) = \boldsymbol{\eta}^T \mathbf{y}, \quad \mathbf{y} = (y_1, \ldots, y_n)^T,$$

with coefficients $\boldsymbol{\eta} = (\eta_i)_{n \times 1}$. The unbiasedness leads to $\boldsymbol{\eta}^T Q = \mathbf{q}^T$, where

$$Q = \{q_\nu(t_i)\}_{n \times l}, \quad \mathbf{q} = (q_1(t_0), \ldots, q_l(t_0))^T,$$

and therefore $\sum_{i=1}^{n} \eta_i q_j(t_i) = q_j(t_0), j = 1, 2, \ldots, l$, which implies

$$\sum_{i=1}^{n} \eta_i \delta(t_i) - \delta(t_0) \in \Lambda_\kappa. \tag{6}$$

Hence, the squared prediction error can be shown to be

$$\mathrm{E}(\hat{Z}(t_0) - Z(t_0))^2 = \sigma^2 \boldsymbol{\eta}^T \boldsymbol{\eta} + \boldsymbol{\eta}^T \Psi \boldsymbol{\eta} - 2\boldsymbol{\eta}^T \boldsymbol{\phi} + \phi(0),$$

where

$$\Psi = \{\phi(t_i - t_j)\}_{n \times n}, \quad \boldsymbol{\phi} = (\phi(t_1 - t_0), \ldots, \phi(t_n - t_0))^T.$$

The goal of universal kriging is to minimize the squared prediction error, subject to the unbiasedness constraints. Letting a vector $\boldsymbol{\rho}$ of $l \times 1$ be the Lagrange multipliers, we need to minimize

$$M(\boldsymbol{\eta}) = \sigma^2 \boldsymbol{\eta}^T \boldsymbol{\eta} + \boldsymbol{\eta}^T \Psi \boldsymbol{\eta} - 2\boldsymbol{\eta}^T \boldsymbol{\phi} + \phi(0) + 2(\boldsymbol{\eta}^T Q - \mathbf{q}^T)\boldsymbol{\rho}.$$

Direct computation finds the universal kriging formula as

$$\begin{cases} (\Psi + \sigma^2 I_n)\boldsymbol{\eta} + Q\boldsymbol{\rho} = \boldsymbol{\phi}, \\ Q^T \boldsymbol{\eta} = \mathbf{q}, \end{cases} \tag{7}$$

where I_n is an $n \times n$ identity matrix.

Next, we show that this universal kriging formula can be interpreted in the content of RKHS. With this IRF$\kappa Z(\cdot)$ and the observed data $\{(t_i, y_i), i = 1 \ldots, n\}, n \geq l$, the smoothing problem is to find a function $f_\alpha(t) \in X_\kappa$ such that it minimizes (Taijeron et al., 1994; Levesley et al., 1999)

$$\sum_{i=1}^{n} (y_i - f(t_i))^2 + \alpha \|f\|_\kappa^2,$$

where $\alpha > 0$ is the smoothing parameter and $\| \cdot \|_\kappa$ is induced by the semi-inner product (4) in Section 2. The minimizer can be shown to be

$$f_\alpha(t) = \sum_{\nu=1}^{l} d_\nu q_\nu(t) + \sum_{i=1}^{n} c_i \phi(t_i - t), \tag{8}$$

where $\mathbf{c} = (c_i)_{n \times 1}$ and $\mathbf{d} = (d_v)_{l \times 1}$ satisfy the following,

$$\begin{cases} (\Psi + \alpha I_n)\mathbf{c} + Q\mathbf{d} = \mathbf{y}, \\ Q^T \mathbf{c} = \mathbf{0}_{l \times 1}. \end{cases}$$

To show the connection between this smoothing formula and universal kriging (7), note that the smoothing formula for an unobserved point $t_0 \in S$ can be rewritten in the following manner

$$f_\lambda(t_0) = (\mathbf{c}, \mathbf{d})^T \begin{pmatrix} \boldsymbol{\phi} \\ \mathbf{q} \end{pmatrix} = (\mathbf{y}^T, \mathbf{0}_{1 \times l}) \begin{pmatrix} \Psi + \alpha I_n & Q \\ Q^T & 0_{l \times l} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\phi} \\ \mathbf{q} \end{pmatrix}$$

$$:= (\mathbf{y}^T, \mathbf{0}_{1 \times l}) \begin{pmatrix} \boldsymbol{\eta}^* \\ \boldsymbol{\rho}^* \end{pmatrix} = \boldsymbol{\eta}^{*T} \mathbf{y},$$

where

$$\begin{pmatrix} \boldsymbol{\eta}^* \\ \boldsymbol{\rho}^* \end{pmatrix} = \begin{pmatrix} \Psi + \alpha I_n & Q \\ Q^T & 0_{l \times l} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\phi} \\ \mathbf{q} \end{pmatrix},$$

or

$$\begin{cases} (\Psi + \alpha I_n)\boldsymbol{\eta}^* + Q\boldsymbol{\rho}^* = \boldsymbol{\phi}, \\ Q^T \boldsymbol{\eta}^* = \mathbf{q}. \end{cases} \tag{9}$$

Remark 3.1. This Eq. (9) is exactly the dual formula of universal kriging (Cressie, 1993; Chilès and Delfiner, 2012). Usually, universal kriging is viewed as a linear estimator of observed data and the smoothing formula is viewed as linear combination of the intrinsic covariance with lower trigonometric trends. From the above discussion, these two views are essentially the same. The connection between universal kriging and the smoothing formula, therefore, is obvious.

Remark 3.2. The smoothing parameter α in (9) and the noise variance σ^2 in (7) play the same role. For example, in the smoothing formula, when α increases to infinity, the minimization procedure demands $\|f\|_\kappa \downarrow 0$, which shows that $c \to 0$, and the smoothing formula reduces to the trigonometric regression (Eubank, 1988). In kriging practice, when σ^2 increases to infinity, the noise overwhelms the spatial dependency, the process becomes essentially uncorrelated. The squared prediction error is dominated by $\sigma^2 \boldsymbol{\eta}^T \boldsymbol{\eta}$. The universal kriging reduces to minimize $\boldsymbol{\eta}^T \boldsymbol{\eta}$ subject to the unbiasedness restriction, which also leads to trigonometric regression prediction. When both α and σ^2 decrease to zero, both smoothing and kriging result in exact interpolation.

Remark 3.3. In this paper, we extend IRF to the circular setting through the RKHS theory. In so doing, we find that the lower monomials in Euclidean spaces need to be replaced by lower trigonometric functions, and the differential operators need to be replaced by low-frequency truncations. The RKHS sheds light into kriging on the circle and provides elementary understanding. This RKHS approach allows us to revisit splines on the circle, see Section 4.1.

4. Examples and discussions

4.1. Spline on the circle

In Craven and Wahba (1979) and Wahba (1990a), the spline on the circle is the minimizer of

$$\sum_{i=1}^n (y_i - f(t_i))^2 + \alpha J(f),$$

where

$$J(f) = \int_S (f^{(m)}(t))^2 dt, \tag{10}$$

This minimizer has been shown to be (Wahba, 1990a)

$$f_\alpha(t) = d + \sum_{i=1}^n c_i R(t_i, t),$$

with

$$R(s, t) = 2 \sum_{n=1}^\infty \frac{1}{n^{2m}} \cos n(s - t).$$

Remark 4.1. This is the smoothing formula (8) for $\kappa = 1$, where the intrinsic covariance function $\phi(\cdot)$ is replaced by $R(s, t)$. Note that the spline kernel $R(s, t)$ is a specified function, for example,

$$m = 1, \qquad R(s, t) = \frac{(s - t)^2}{2} - \pi |s - t| + \frac{\pi^2}{3},$$

$$m = 2, \qquad R(s, t) = -\frac{(s - t)^4}{24} + \frac{\pi |s - t|^3}{6} - \frac{\pi^2 (s - t)^2}{6} + \frac{\pi^4}{45}.$$

These kernels $R(s, t)$ are clearly positive definite (Schoenberg, 1942) on the circle, and are valid covariance functions.

Remark 4.2. In Euclidean spaces, the order m in (10) plays a significant role. It indicates the smoothness assumption of the function, and relates to the order of IRF in kriging (Kent and Mardia, 1994). However, for this spline on the circle, this order m only alters the covariance functions (see Remark 4.1), and loses its connection to the order κ of IRF. Therefore, the spline with derivative penalty has limited application for circular processes. As shown in Lemma 2.1, the low-frequency truncation operation shall be used, leading to the more appropriate spline model on the circle

$$\frac{1}{n} \sum_{i=1}^{n} (y_i - f(t_i))^2 + \alpha \|f\|_\kappa^2, \tag{11}$$

where $\| \cdot \|_\kappa$ is induced by the semi-inner product (4) in Section 2. A more general approach for hyperspheres can be found in Taijeron et al. (1994).

4.2. Splines and kriging

The spline model (11) is exactly the same as the smoothing formula in Section 3, where the equivalence between the smoothing formula and kriging is discussed. It is clear that this spline (11) and kriging are also equivalent on the circle. The RKHS theory on the circle offers a clear view of this connection. In addition, following Remark 3.1, spline is a linear combination of the intrinsic covariance functions with lower trigonometric trends, while universal kriging is a linear estimator of observed data. They arrive at the same conclusion as dual formulations of kriging (Cressie, 1993).

The connections between splines and kriging have been extensively discussed in literature, including Matheron (1981), Watson (1984), Lorenc (1986), Cressie (1989, 1990, 1993), Wahba (1990a,b), Kent and Mardia (1994), Laslett (1994), Furrer and Nychka (2007) among others. Furrer and Nychka (2007) show that, given a covariance function, one can construct a reproducing kernel and obtain a general spline estimate in Euclidean space. In this paper, we show this connection formally for circular processes using the IRFs.

4.3. Ordinary kriging

The ordinary kriging is well known and has been widely used in a variety of spatial analysis contexts. On the circle, ordinary kriging is the universal kriging developed in Section 3 with $\kappa = 1$. The process $Z(t)$ is IRF1 with mean and covariance

$$\mathrm{E}(Z(t)) = \beta_1, \qquad \mathrm{cov}(Z(x), Z(y)) = H(x, y) = \phi(x - y) - \phi(x - \tau_1) - \phi(y - \tau_1) + \phi(0) + 1.$$

The process is not stationary, but a direct computation gives $\mathrm{var}(Z(x) - Z(y)) = 2\phi(0) - 2\phi(x - y)$. That is, the variance of the difference at two points only depends on their circular distance and we can define

$$\tau(\theta) := \phi(0) - \phi(\theta) = \sum_{n=1}^{\infty} \gamma_n (1 - \cos n\theta),$$

as the semi-variogram (Schoenberg, 1942; Huang et al., 2011). The process can be viewed as intrinsically stationary on the circle. Note that $\phi(\theta)$ relates to the semi-variogram directly through (Huang et al., 2011)

$$\phi(\theta) = c_0 - \tau(\theta), \quad c_0 \geq \frac{1}{\pi} \int_0^\pi \tau(\theta) d\theta.$$

Therefore, with $\boldsymbol{\tau} = (\tau(t_1 - t_0), \ldots, \tau(t_n - t_0))^T$, $\mathbf{1}_n = (1, 1, \ldots, 1)^T$, and $\Gamma = \{\tau(t_i - t_j)\}$, we have the kriging estimator

$$f_\alpha(t_0) = \boldsymbol{\eta}^T \mathbf{y},$$

where $\boldsymbol{\eta}$ satisfies

$$-\Gamma \boldsymbol{\eta} + \rho \mathbf{1}_n = \boldsymbol{\tau}, \quad \mathbf{1}_n^T \boldsymbol{\eta} = 1.$$

This is exactly the ordinary kriging in the spatial literature (Cressie, 1993).

4.4. Brownian bridge

The Brownian bridge $\{B(t), t \in S\}$ is a random process on the circle with mean zero and

$$\text{cov}(B(s), B(t)) = 2\pi \min\{s, t\} - st, \quad s, t \in S.$$

It is clearly not a stationary process. However, for any allowable measure $\lambda = \sum_{i=1}^{m} \lambda_i \delta(t_i) \in \Lambda_1$ (hence $\sum_{i=1}^{m} \lambda_i = 0$),

$$\text{cov}(B(\iota_t\lambda), B(\iota_s\lambda)) = \sum_{i,j=1}^{m} \lambda_i\lambda_j\{2\pi \min\{t_i + t, t_j + s\} - (t_i + t)(t_j + s)\}$$

$$= -\pi \sum_{i,j=1}^{m} \lambda_i\lambda_j|t_i + t - t_j - s| + (1/2) \sum_{i,j=1}^{m} \lambda_i\lambda_j(t_i + t - t_j - s)^2,$$

where the last equality is based on $\min\{a, b\} = \frac{1}{2}((a + b) - |a - b|), a > 0, b > 0$ and $\sum_{i=1}^{m} \lambda_i = 0$. That is, $B(\iota_t\lambda)$ is stationary with respect to t, showing that the Brownian bridge is an IRF1.

On the other hand, one can consider the Fourier series representation of the Brownian bridge

$$B(t) = B_0 + \sum_{n=1}^{\infty} (B_{n,c} \cos(nt) + B_{n,s} \sin(nt)).$$

Direct computation shows that for all $n, m \geq 1$, $\text{E}(B_{n,c}B_{m,c}) = \text{E}(B_{n,s}B_{m,s}) = \frac{2}{n^2}I(m, n)$ and $\text{E}(B_{n,c}B_{m,s}) = 0$. Therefore, the truncated process $B_1(t) = \sum_{n=1}^{\infty}(B_{n,c} \cos(nt) + B_{n,s} \sin(nt))$ is clearly stationary, and so the Brownian bridge is an IRF1 by Lemma 2.1. In addition, its intrinsic covariance function is given by $\phi(s - t) = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n(s - t))$, which is exactly the same as the $R(s, t)$ with $m = 1$ in Section 4.1. However, we have $\text{E}(B_0B_{n,c}) = -\frac{2}{n^2}, n \geq 1$, that is, B_0 is correlated with all $B_{n,c}, n \geq 1$. Such a coupling reveals that the Brownian bridge is not stationary. But, by truncation this coupling is removed, and the truncated process becomes stationary.

Acknowledgments

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